A novel method for determining the period of the mathematical pendulum at large amplitudes and a discussion of its applicability to other selected oscillations in the plane

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Summary of the present paper

The period of point masses moving along curved paths of oscillation in the vertical plane – such as the mathematical pendulum and the cycloidal pendulum – is to be calculated in the present paper in a simpler way than has previously been the case.

Most of these periods, when the point mass is farther away from the equilibrium position, that is, from its lowest point on the path of oscillation, depend on that distance, which in general makes the calculation more difficult.

In order to calculate a period independent of this distance, namely the amplitude, one needs to know, apart from the mass, only the directive quantity, the quotient of restoring force and the corresponding displacement or directive path.

In this paper, this knowledge is extended to periods that depend on their amplitude and is applied to them.

In comparison with the customary methods, this leads in part to differences – in the case of the mathematical pendulum by at most one per cent at a deflection of 90° – and in part, as in the case of the cycloidal pendulum, not at all. For the elliptical pendulum, which was likewise investigated, no references in the literature could be found.

In the experimental part, the calculated period of the mathematical pendulum could be confirmed, taking account of sources of error and measurement accuracy.

The author comes to the conclusion that the independently developed method could indeed find application in physics.

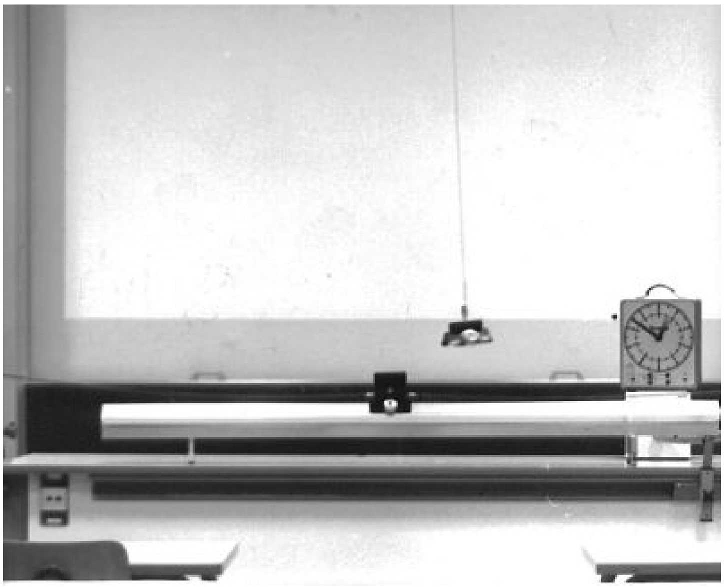

Fig. 1

Contents

1. Introduction

1.1. Motivation and basic idea

The period of the mathematical pendulum at large amplitudes has hitherto been determined by rather difficult methods of infinitesimal calculus and series expansion. The new method developed in the present paper for determining the periods of selected oscillations in the plane is intended, on the one hand, to simplify the calculation and, on the other hand, to be compared with the old method in respect of its accuracy by means of two examples.

The basic idea is the simultaneous simulation of the respective pendulum motion by a rectilinear harmonic oscillation (spring pendulum, oscillator).

1.2. Assumptions

All selected oscillations proceed from a point mass which is to oscillate in the vertical plane on a massless string about a fixed point.

In this connection, the pendulum string may be deflected by at most 90° from the equilibrium position, since the maximum restoring force must lie at the initial height, where no additional acceleration is to occur.

In order to prevent free fall, rigid rods would otherwise have to be used instead of strings, but these could not, as required, be fitted to the evolutes of the paths of oscillation. Despite such fitting, the length of the pendulum in the equilibrium position is assumed for the sake of simplification.

The radius of curvature of the path of oscillation must either remain constant from the equilibrium position onwards or decrease continuously, so that free fall is likewise prevented.

Frictional and damping effects are to be neglected.

Otherwise, further calculations for the period would be necessary, but these would go beyond the scope of this paper.

2. Main part

2.1. Theoretical solution of the problem

2.1.1. General solution

The period of a harmonic oscillation is given by:

$$T = 2\pi\sqrt{\frac{m}{D}}. \tag{a}$$

According to the law of conservation of energy,

$$\frac{1}{2}mv^2 = \frac{1}{2}Ds^2 = mgh. \tag{b}$$

In an oscillation, the potential energy W_{pot} and the kinetic energy W_{kin} form the constant sum:

$$W_{\text{pot}} + W_{\text{kin}} = W_{\text{pot},0}. \tag{c}$$

$W_{\text{pot},0}$ is now to be expressed by the elastic energy W_{Sp} in order to determine a directive quantity $D(\varphi_0)$ depending on the amplitude. This $D(\varphi_0)$ is then to be inserted into formula (a) in order to determine the period.

For this purpose, an equivalent motion of the mass m under the force F along the directive path s is required. One has:

$$D = \frac{F}{s} \qquad \text{(Hooke's law)} \tag{d}$$

and

$$F = ma = ml\,\ddot{\varphi} \qquad \text{(Newton)}. \tag{e}$$

In what follows, repeated reference will be made to the directive quantity D of the planar string pendulum at minimal amplitude. In this case, for the restoring force F_R, which is tangential to the path of oscillation, the normal force is cancelled by the tension of the string, so that:

$$F_R = mg\sin\varphi = -ml\,\ddot{\varphi}. \tag{f}$$

Hence,

$$\ddot{\varphi} = \sin\varphi\,\frac{g}{l}. \tag{g}$$

For very small angles of deflection φ, $\sin(\varphi)$ may be replaced by φ, and the conditions for harmonic oscillation are satisfied. Therefore,

$$\ddot{\varphi} = \omega^2\varphi \tag{h}$$

and from (g) it follows that

$$\omega = \sqrt{\frac{g}{l}}. \tag{i}$$

Thus,

$$D = m\omega^2 = m\frac{g}{l}. \tag{j}$$

For larger amplitudes, the directive quantity D must be composed of the product of the directive quantity for harmonic oscillation and a functional value $f(\varphi_0)$ of the angle of deflection φ_0 which determines the non-harmonic oscillation. One writes:

$$D(\varphi_0) = \left(m\frac{g}{l}\right) f(\varphi_0). \tag{k}$$

However, the functional value $f(\varphi_0)$ is not defined for the harmonic condition of the energy theorem, so that it must cancel when introduced. With $s = xl$, one has:

$$\frac{1}{2}\frac{\frac{f(\varphi_0)mxg}{xl}(xl)^2}{f(\varphi_0)} = mgh. \tag{l}$$

In this way, the unchanging directive path s can be calculated from the energy theorem if the functional value $f(\varphi_0)$ is assigned to the restoring force F_R. With

$$\frac{1}{2}\left(m\frac{g}{l}\right)s^2 = mgh$$

one obtains for the directive path:

$$s = \sqrt{2lh}. \tag{m}$$

The restoring force F_R must therefore be developed independently of the energy theorem. Rather, it is determined by the initial oscillation as the force acting tangentially to the path of oscillation at the initial height and thus being maximally restoring. From

$$F_R = -ma \tag{n}$$

one obtains, with (a) and (d), the period:

$$T = 2\pi\sqrt{\frac{\sqrt{2lh}}{a}}. \tag{o}$$

Geometrically, the directive path s in (m) is the chord of a circle of radius l. Since this circle is the osculating circle of the path of oscillation at the equilibrium position, this chord must lead from a point of the circle at the initial height h to the equilibrium position.

The simplest form of a path of oscillation which coincides with the osculating circle and possesses a point-like evolute is that of the mathematical pendulum (Fig. 1). It is therefore treated first.

2.1.2. The mathematical pendulum

From Fig. 1 one obtains for the directive path s:

$$s = 2\sin\left(\frac{\varphi_0}{2}\right)l.$$

With (f), the maximally restoring force F_R is:

$$F_R = -mg\,\sin(\varphi_0).$$

Hence, by (a) and (d), the period is:

$$T = 2\pi\sqrt{\frac{l}{\cos\left(\frac{\varphi_0}{2}\right)g}}.$$

2.1.3. The cycloidal pendulum

For the common cycloid (Fig. 2), the parametric equation is:

$$x = a(\Phi - \sin\Phi) \qquad \text{and} \qquad y = a(\cos\Phi - 1).$$

For the radius of curvature ρ, one obtains:

$$\rho = 4a\sin\left(\frac{\phi}{2}\right).$$

Since the angles Φ increase from the initial height h to the equilibrium position at $\Phi = \pi$, or respectively since the sine of $\Phi/2$ decreases again after passing the equilibrium position on the way to the turning point, the assumptions for calculating the period are fulfilled.

The initial height is:

$$h = 2a + y = a(\cos \Phi_0 + 1).$$

If h is inserted into (m), then with $l = 4a$ one has:

$$s = l\sqrt{\frac{\cos \phi_0 + 1}{2}} = \cos\left(\frac{\phi_0}{2}\right) l.$$

Since the tangential acceleration b_t always acts perpendicularly to the radius of curvature ρ, one obtains:

$$b_t = \alpha \cos\left(\frac{\phi_0}{2}\right) g.$$

If $\Phi_0 = 0$, then $b_t = g$, and if $\Phi_0 = \pi$, then $b_t = 0$. Therefore the multiplier must be $\alpha = 1$, and so:

$$b_t = \cos\left(\frac{\phi_0}{2}\right) g.$$

Finally, from (o) one obtains for the period:

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

This had already been proved by Huygens centuries earlier.

2.1.4. The elliptical pendulum

For the ellipse (Fig. 3), the equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

For the radius of curvature ρ, differentiation yields (see Appendix):

$$\rho = \frac{\left(a^4 - a^2x^2 + b^2x^2\right)^{3/2}}{a^4 b}.$$

From the equilibrium position with $x = 0$ towards the turning points, x increases, and since $a^2x^2 > b^2x^2$, the radius of curvature decreases continuously. At the equilibrium position, therefore,

$$\rho = l = \frac{a^2}{b}.$$

With

$$y = -b\sqrt{1 - \frac{x^2}{a^2}}$$

the initial height h is:

$$h = b + y = b\left(1 - \sqrt{1 - \frac{x^2}{a^2}}\right).$$

Hence, by (m), the directive path s is:

$$s = \sqrt{2a\left(a - \sqrt{a^2 - x^2}\right)}.$$

Thus, the directive path is independent of the semi-minor axis of the ellipse.

The tangential acceleration b_t is (according to Fig. 3):

$$b_t = \cos\beta\, g = \frac{bxg}{\sqrt{a^4 - a^2x^2 + b^2x^2}}.$$

Using (o), one obtains for the period:

$$T = 2\pi\sqrt{\frac{\sqrt{a^4 - a^2x^2 + b^2x^2}\,\sqrt{2a\left(a - \sqrt{a^2 - x^2}\right)}}{bxg}}.$$

For $x = 0$, the period is admittedly not defined, but for very small x one has:

$$T = 2\pi\sqrt{\frac{a^2}{bg}} = 2\pi\sqrt{\frac{l}{g}}.$$

For $x = a$, the following statement is obtained:

> The period of oscillation on an ellipse at maximum deflection is independent of the semi-minor axis.

2.2. Experiment on the mathematical pendulum

In the experimental part, the investigation was restricted to the mathematical pendulum for reasons of time.

On the one hand, the construction of evolutes would have been very laborious; on the other hand, the cycloidal and elliptical oscillations would have introduced additional sources of error. These include the difficulty of making the pendulum move exactly along the evolute, since after only a few oscillations the pendulum leaves the vertical plane. In addition, impulsive and frictional forces arise both when setting the pendulum in motion and in the bearing, and at large amplitudes these make the respective pendulum tend towards the mathematical pendulum.

The latter may therefore be regarded as a part standing for the whole.

2.2.1. Experimental set-up

Two rods, each one metre long, are connected by a third rod and four clamps and are passed symmetrically through two ceiling hooks. In the middle, a protractor and the suspension for the pendulum string of length 1.60 m are additionally attached.

An air track is placed parallel to this on the laboratory table in such a way that its centre lies vertically beneath the tip of the protractor.

A trolley of mass 0.2 kg placed at the centre is held by two springs in such a way that between the fixing at the end of the laboratory table and the trolley there is a directive quantity of 2 N/m.

A stopwatch is placed within reach at the initial height of both the pendulum and the trolley.

Figures relating to the experiment are given in the Appendix. All the apparatus used is also listed there.

2.2.2. Experimental procedure and results

First, the independence of the period from the pendulum mass was confirmed: once, a ball of approximately 5 kg mass was attached to the pendulum string, and on another occasion the trolley, loaded to a mass of 1 kg, was attached. The difference between the measured periods amounted to one hundredth of a second for each set of ten periods.

In the simulation of the mathematical pendulum by the harmonic oscillator, the time difference per oscillation was 0.6 s (to and fro) for ten oscillations and a maximum pendulum deflection of 60°.

To calculate the mass of the harmonic oscillator, the formula

$$m = \frac{2lD}{\cos\left(\frac{\varphi_0}{2}\right) g}.$$

was used. This can be obtained directly from the periods of the two systems.

Here, the directive quantity D_0 for the oscillator is twice the spring constant of either of the two identical tension springs used. The period of the oscillator is therefore:

$$T = 2\pi\sqrt{\frac{m}{2D}}.$$

The period of the mathematical pendulum calculated above is:

$$T = 2\pi\sqrt{\frac{l}{\cos\left(\frac{\varphi_0}{2}\right) g}}.$$

The damping of the pendulum at a deflection of 60° amounted to 1°, that of the oscillator to 0.01 m at a deflection of 0.9 m.

The measured values for deflections of the mathematical pendulum from 0° to 90° are given in Table 1 in the Appendix.

2.3. Error calculations

The period of the mathematical pendulum has hitherto been given in terms of the elliptic integral

$$\int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - k^2 \sin^2 \phi}}$$

with

$$k = \sin\left(\frac{\varphi_0}{2}\right)$$

as

$$T = 2\pi\sqrt{\frac{l}{g}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \dots\right)}.$$

In order to permit a direct comparison with the new formula for the period,

$$T = 2\pi\sqrt{\frac{l}{\cos\left(\frac{\varphi_0}{2}\right)g}},$$

the functional value

$$f(\varphi_0) = \sqrt{\frac{1}{\cos\left(\frac{\varphi_0}{2}\right)}}$$

is transformed into the comparison series

$$1 + \left(\frac{k}{2}\right)^2 + \left(\frac{1\cdot 5}{1\cdot 2}\right)^2\left(\frac{k}{2}\right)^4 + \left(\frac{1\cdot 5\cdot 9}{1\cdot 2\cdot 3}\right)^2\left(\frac{k}{2}\right)^6 + \ldots$$

with the help of the series for $|x| < 1$,

$$(1+x)^p = 1 + px + \frac{p(p-1)x^2}{1\cdot 2} + \frac{p(p-1)(p-2)x^3}{1\cdot 2\cdot 3} + \ldots$$

and the relation

$$f(\varphi_0) = \frac{1}{\sqrt[4]{1-\sin^2\left(\frac{\varphi_0}{2}\right)}}$$

at

$$x = -\sin^2\left(\frac{\varphi_0}{2}\right) = -k^2.$$

It follows from this that the two series agree up to the first quadratic term.

If the old series is divided by the new one and 1 is subtracted, the smaller in magnitude of the two relative error series is obtained:

$$\frac{1}{64}k^4 + \frac{1}{64}k^6 + \frac{231}{16384}k^8 + \ldots$$

If, conversely, one divides in the opposite order and again subtracts 1, the relative error is larger:

$$\frac{1}{64}k^4 + \frac{1}{64}k^6 + \frac{235}{16384}k^8 + \ldots$$

The series for the absolute error is obtained by subtracting the old series from the new one:

$$\frac{1}{64}k^4 + \frac{5}{256}k^6 + \frac{335}{16384}k^8 + \ldots$$

The values for selected angles from 0° to 90° for all the series listed here are recorded in tables in the Appendix.

2.4. Discussion

At this point, let us reflect for a moment on the proportionalities occurring in the formula

$$T = 2\pi\sqrt{\frac{\sqrt{2lh}}{a}}.$$

First of all, it must be noted that the directive path s is not proportional to the acceleration a of the restoring force F_R; rather, the functional value $f(\varphi_0)$ contained in a must be taken into account.

The length of the path of oscillation is adjusted via the directive path: if the initial height h is very small, then the distance to be covered in the y-direction along the path of oscillation, and hence also the period, becomes smaller.

If the path of oscillation is also very flat, the small initial height is offset by a large pendulum length, so that one obtains a moderate value for the length of the directive path, as is particularly evident in the case of maximum amplitudes of the elliptical pendulum.

For strongly curved paths of oscillation and large initial heights, as in the mathematical pendulum, the period is relatively larger.

In the immediate neighbourhood of the equilibrium position, the numerical values of directive path and acceleration cancel each other, so that the directive path becomes the pendulum length l and the acceleration becomes the acceleration due to gravity g.

Strictly speaking, the period is not defined at the equilibrium position itself, because there both the initial height h and the acceleration a become zero.

The greater the acceleration a, the smaller the period; otherwise the opposite relation holds.

These findings could be confirmed reasonably well in the experiment on the mathematical pendulum.

3. Conclusion

Looking back, it may be said that the actual procedure did not entirely coincide with the one presented here for reasons of clarity.

Thus, the problem of the mathematical pendulum was solved first, and only afterwards was a general solution of the problem formulated. All periods were calculated without knowledge of the periods given in the corresponding literature; access to those results, like the experiment itself, followed only weeks later.

The calculation could indeed be greatly simplified, and in respect of accuracy the error limit of 0.01 or one per cent relative to the old method did not need to be exceeded.

Here, however, the elliptical pendulum could not be included, since no references in the literature concerning its period could be found.

Most probably, however, the error limit will shift upwards, since the mathematical pendulum, as a special case of the elliptical pendulum whose period can be transformed into that of the mathematical pendulum when the semi-axes are equal in length, already shows a deviation of almost one per cent at maximum amplitude with regard to its period.

The agreement of the period of the cycloidal pendulum with the statements in the literature is, however, beyond contradiction.

3.1. Critical appraisal of the paper

Without doubt, the evolute problem has not yet been fully solved. If one no longer assumes the length of the pendulum in the equilibrium position, it may under certain circumstances once again be necessary to resort to methods of infinitesimal calculus and series expansion in order to obtain greater accuracy.

Against this, however, stand, on the one hand, the mathematical pendulum and, on the other hand, the cycloidal pendulum.

In the former case, a pronounced evolute is absent despite the deviation; in the latter, no deviation exists despite the presence of an evolute.

Thus, the new method must be regarded as independent.

A more precise clarification of the matter would be provided by the measurements of further periods beyond that of the mathematical pendulum which are unfortunately lacking in this paper. However, very exacting demands would have to be made on measurement accuracy and on the execution of the experiments.

Taken as a whole, the method developed in this paper does require a certain change in manner of thinking; nevertheless, in view of the good results obtained, it ought to be capable of finding application in physics.

4. Appendix

Figures

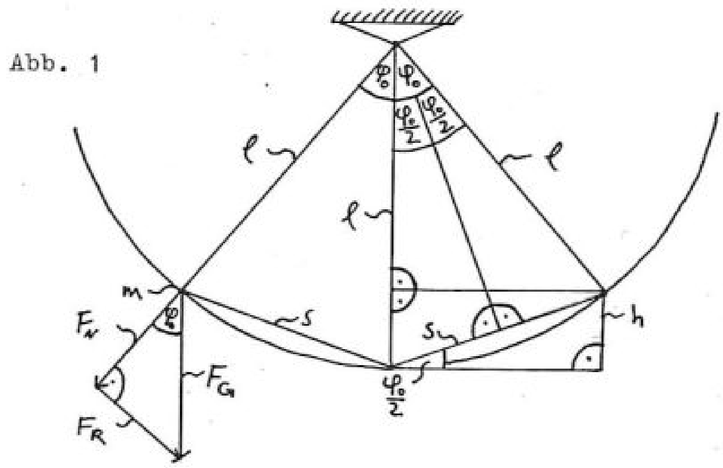

Fig. 2

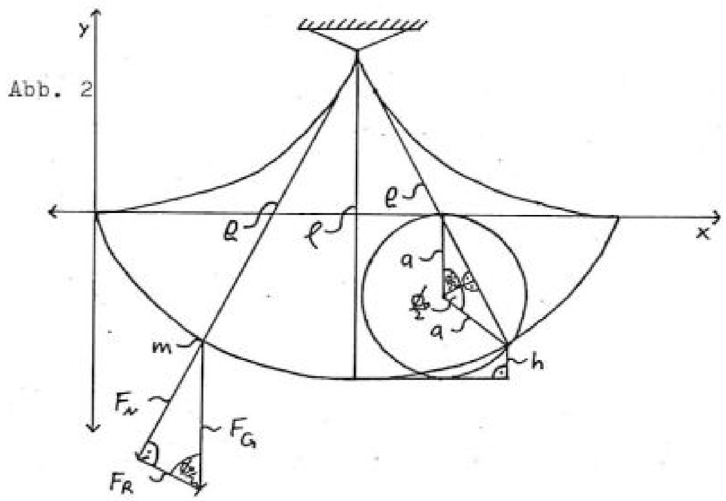

Fig. 3

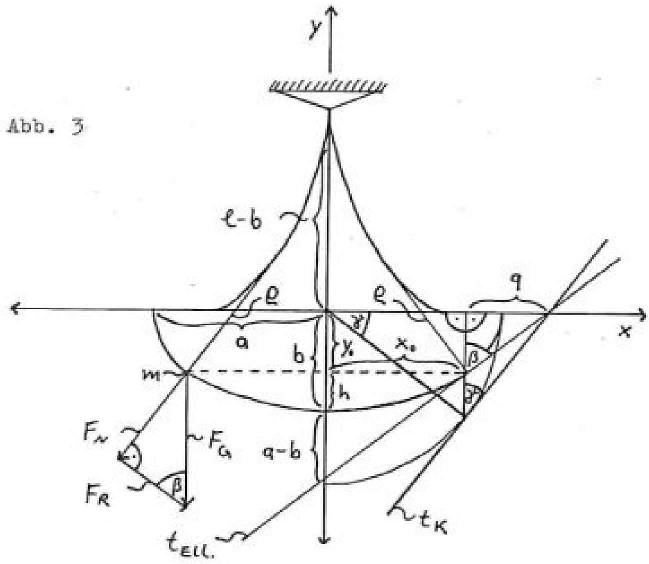

Fig. 4

All constructions were carried out with the aid of (9).

Supplementary calculations

For the radius of curvature, the general formula is:

$$\rho = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}}.$$

The common cycloid has the parametric equation

$$x = a(\Phi - \sin \Phi) \qquad \text{and} \qquad y = a(\cos \Phi - 1).$$

Hence, for $(\dot{x}^2 + \dot{y}^2)^{3/2}$ one may write

$$\left(a^2 - 2\cos\phi\, a^2 + \cos^2\phi\, a^2 + \sin^2\phi\, a^2\right)^{3/2} = \left(2a^2 - 2\cos\phi\, a^2\right)^{3/2}.$$

With

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = (a - \cos\phi\, a)(-\cos\phi\, a) - (-\sin\phi\, a)\sin\phi\, a = a^2 - \cos\phi\, a^2$$

one therefore obtains for the radius of curvature

$$\rho = 2a\sqrt{2(1 - \cos\phi)} = 4a \sin\left(\frac{\phi}{2}\right).$$

If, for the ellipse, one temporarily writes $x = \sin(\varphi)a$ and

$$y = -b\sqrt{1 - \frac{x^2}{a^2}} = -\cos\varphi\, b,$$

then one obtains

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = \cos\varphi\, a(\cos\varphi\, b) - \sin\varphi\, b(-\sin\varphi\, a) = ab.$$

Thus, the radius of curvature is:

$$\rho = \frac{(\cos^2\varphi\, a^2 + \sin^2\varphi\, b^2)^{3/2}}{ab}.$$

If one then reverses the transformation and multiplies the fraction by a^3, one obtains for ρ:

$$\rho = \frac{(a^4 - a^2x^2 + b^2x^2)^{3/2}}{a^4 b}.$$

The evolute of the common cycloid is likewise a cycloid with the same a.

For the evolute of the ellipse, however, one has:

$$\xi = \frac{(a^2 - b^2)\cos^3 t}{a} \qquad \text{and} \qquad \eta = \frac{(a^2 - b^2)\sin^3 t}{b}$$

with $x = a\cos t$ and $y = b\sin t$.

In order to calculate the tangential acceleration b_t of the ellipse, the following preliminary considerations are necessary:

$$\tan\gamma = \frac{\sqrt{a^2 - x^2}}{x} = \frac{q}{\sqrt{a^2 - x^2}}.$$

It follows directly that

$$q = \frac{a^2 - x^2}{x}.$$

For

$$\cos\beta = \frac{y}{\sqrt{q^2 + y^2}}$$

one may write, after simplification,

$$\cos\beta = \frac{bx}{\sqrt{a^4 - a^2x^2 + b^2x^2}}.$$

In order to transform the formula for the period of the elliptical pendulum,

$$T = 2\pi\sqrt{\frac{\sqrt{a^4 - a^2x^2 + b^2x^2}\,\sqrt{2a(a - \sqrt{a^2 - x^2})}}{bxg}},$$

into that of the mathematical pendulum, one must set $x = \sin(\varphi_0)a$ and $a = b = 1$. Thus,

$$T = 2\pi\sqrt{\frac{a^2\sqrt{2\left(1 - \sqrt{1 - \frac{x^2}{a^2}}\right)}}{xg}} = 2\pi\sqrt{\frac{l\sqrt{2(1 - \cos\varphi_0)}}{\sin\varphi_0\, g}}.$$

Hence, finally,

$$T = 2\pi \sqrt{\frac{l}{\cos\left(\frac{\varphi_0}{2}\right) g}}.$$

Experimental apparatus

- 2 Table clamps for the laboratory table with two holders for the springs
- 2 Tension springs, each 1 m long, with a spring constant of approximately 2 N/m and a diameter of approximately 0.01 m
- 1 Air track, 2 m long, with air regulator
- 2 Trolleys with vane and outer hook, each of mass 0.2 kg
- 1 Spherical mass of approximately 5 kg
- 1 Pendulum string of at most 1.60 m in length
- 1 Stopwatch
- 2 Ceiling hooks
- 1 Stand extendable up to 2 m with holder
- 7 Clamps
- 7 Rods of lengths 1 m twice, 0.6 m once, 0.5 m twice, 0.1 m once, and 0.2 m once as an angle rod
- 1 Clamp for protractor with small wooden piece
- 1 Protractor
- 1 Adhesive tape

as well as various weights used as loads.

Tables

Table 1

φ_0/degrees	T_{exp}/s	T_{thn}/s	T_{tha}/s	ΔT_{an}/%	ΔT_{aa}/%
90	2.95	3.02	3.00	2.26	1.52
80	2.87	2.90	2.89	1.02	0.58
70	2.78	2.80	2.80	0.86	0.61
60	2.72	2.73	2.72	0.26	0.13
50	2.65	2.67	2.66	0.60	0.54
40	2.61	2.62	2.62	0.31	0.29
30	2.58	2.58	2.58	0.09	0.08
20	2.55	2.56	2.56	0.29	0.29
10	2.52	2.54	2.54	0.90	0.90
5	2.51	2.54	2.54	1.15	1.15

T_{exp} = experimentally determined period (mean values)
T_{thn} = theoretical period according to the new method
T_{tha} = theoretical period according to the old method
ΔT_{an} = absolute difference between T_{exp} and T_{thn} with T_{thn} as reference value
ΔT_{aa} = absolute difference between T_{exp} and T_{tha} with T_{tha} as reference value

Here, $l = 1.60$ m and $g = 9.80665$ m/s^2.

Table 2

φ_0/degrees	$f(T_0)_n$	$f(T_0)_a$	$\Delta f(T_0)_{rkl}$	$\Delta f(T_0)_{rgr}$	$\Delta f(T_0)_a$
90	1.18921	1.18034	0.74558	0.75118	0.88665
80	1.14254	1.13749	0.44214	0.44411	0.50517
70	1.10489	1.10214	0.24815	0.24877	0.27418
60	1.07457	1.07318	0.12916	0.12933	0.13879
50	1.05042	1.04978	0.06045	0.06049	0.06350
40	1.03159	1.03134	0.02418	0.02418	0.02494
30	1.01748	1.01741	$8 \cdot 10^{-3}$	$8 \cdot 10^{-3}$	$8 \cdot 10^{-3}$
20	1.00768	1.00766	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$	$1 \cdot 10^{-3}$
10	1.00191	1.00191	$1 \cdot 10^{-4}$	$1 \cdot 10^{-4}$	$1 \cdot 10^{-4}$
5	1.00048	1.00048	$6 \cdot 10^{-6}$	$6 \cdot 10^{-6}$	$6 \cdot 10^{-6}$
4	1.00030	1.00030	$3 \cdot 10^{-6}$	$2 \cdot 10^{-6}$	$2 \cdot 10^{-6}$
3	1.00017	1.00017	$8 \cdot 10^{-7}$	$7 \cdot 10^{-7}$	$8 \cdot 10^{-7}$
2	1.00008	1.00008	$3 \cdot 10^{-7}$	$2 \cdot 10^{-7}$	$2 \cdot 10^{-7}$
1	1.00002	1.00002	$2 \cdot 10^{-7}$	$1 \cdot 10^{-7}$	$1 \cdot 10^{-7}$
0	1.00000	1.00000	0	0	0

$f(T_0)_n$ = function value of the period of the mathematical pendulum by which the period of the cycloidal pendulum must be multiplied, according to the new method
$f(T_0)_a$ = the same according to the old method
$\Delta f(T_0)_{rkl}$ = smaller relative difference of the two
$\Delta f(T_0)_{rgr}$ = larger relative difference of the two
$\Delta f(T_0)_a$ = absolute difference of the two

Here, all $\Delta f(T_0)_x$ are given in per cent.

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Postscript

Today, of course, I am in a position to calculate the correct periods and to specify the error of my approximation formula.

It holds that:

$$mgh_0 = mgh + \frac{1}{2}m\frac{\mathrm{d}s^2}{\mathrm{d}t^2}.$$

With $x = x(t)$ and $y = y(t)$, one has:

$$T = \sqrt{\frac{8}{g}}\int_0^{s_0}\sqrt{\frac{1}{h_0 - h}}\,\mathrm{d}s = \sqrt{\frac{8}{g}}\int_0^{x_0}\sqrt{\frac{1 + (y')^2}{y_0 - y}}\,\mathrm{d}x = \sqrt{\frac{8}{g}}\int_0^{t_0}\sqrt{\frac{\dot{x}^2 + \dot{y}^2}{y_0 - y}}\,\mathrm{d}t.$$

The approximation formula is:

$$T = 2\pi\sqrt{\frac{\sqrt{2lh}}{a}} = 2\pi\sqrt{\frac{\sqrt{2\left(1+y'(0)^2\right)^{3/2}\,y_0\,(1+y_0'^2)}}{g y_0'\sqrt{y''(0)}}} = 2\pi\sqrt{\frac{\sqrt{2\left(\dot{x}(0)^2+\dot{y}(0)^2\right)^{3/2}\,y_0\,(\dot{x}_0^2+\dot{y}_0^2)}}{g\dot{y}_0\sqrt{\dot{x}(0)\ddot{y}(0)-\dot{y}(0)\ddot{x}(0)}}}.$$

With $x = a\sin\varphi$, $y = b - b\cos\varphi$,

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}\ \mathrm{d}\varphi, \qquad k = \sin\left(\frac{\varphi_0}{2}\right)$$

and

$$\varepsilon^2 = \frac{a^2 - b^2}{a^2}$$

one obtains for the approximation formula, with $\varepsilon^2 < 1$,

$$T = \frac{2\pi a\sqrt[4]{1-\varepsilon^2\sin^2\varphi_0}}{\sqrt{bg\cos\left(\frac{\varphi_0}{2}\right)}}$$

the series expansion

$$T = \frac{2\pi a}{\sqrt{bg}}\left\{1 + \frac{1}{4}k^2 - \varepsilon^2k^2 + \frac{5}{32}k^4 + \frac{3}{4}\varepsilon^2k^4 - \frac{3}{2}\varepsilon^4k^4 + \mathcal{O}(k^6)\right\}.$$

For the elliptical pendulum, one has exactly

$$T = \sqrt{\frac{8a^2}{bg}}\int_0^{\varphi_0}\sqrt{\frac{1-\varepsilon^2\sin^2\varphi}{\cos\varphi - \cos\varphi_0}}\ \mathrm{d}\varphi$$

with the series expansion

$$T = \frac{2\pi a}{\sqrt{bg}}\left\{1 + \frac{1}{4}k^2 - \varepsilon^2k^2 + \frac{9}{64}k^4 + \frac{3}{8}\varepsilon^2k^4 - \frac{3}{4}\varepsilon^4k^4 + \mathcal{O}(k^6)\right\}.$$

Using the quadratically convergent arithmetic–geometric mean $M\Big(1, \cos\left(\frac{\varphi_0}{2}\right)\Big)$, one obtains an even more accurate formula, and one which is exact for the mathematical pendulum (because $a = b$, that is, $\varepsilon = 0$):

$$T = \frac{2\pi a}{\sqrt{bg}}\left\{\frac{1}{M\Big(1,\cos\left(\frac{\varphi_0}{2}\right)\Big)} - \varepsilon^2k^2 + \frac{3}{8}\left(\varepsilon^2 - 2\varepsilon^4\right)k^4 + \frac{5}{64}\left(\varepsilon^2 + 12\varepsilon^4 - 16\varepsilon^6\right)k^6 + \mathcal{O}\Big(\varepsilon^2k^8\Big)\right\}.$$

With $e := \varepsilon^2$, the expression in braces is

$$g(k,e) = \frac{2}{\pi}\int_0^{\pi/2}\sqrt{\frac{1}{1-k^2\sin^2x} - 4e\,k^2\sin^2x}\ \mathrm{d}x$$

and, via $t := k^2 \sin^2 x$, one obtains automatically:

$$g(k,e) = \sum_{n\geq 0} c_{2n}(e) k^{2n}, \qquad c_{2n+1}(e) = 0.$$

Then

$$\sqrt{\frac{1}{1-t} - 4et} = \frac{\sqrt{1-4et+4et^2}}{\sqrt{1-t}} = \left(\sum_{m\geq 0} b_m(e) t^m\right)\left(\sum_{j\geq 0} a_j t^j\right)$$

with

$$a_j = \frac{\binom{2j}{j}}{4^j} \qquad \text{(series of } (1-t)^{-1/2}\text{)}$$

and $b_m(e)$ as the coefficients of the square-root series

$$\sqrt{1-4et+4et^2} = \sum_{m\geq 0} b_m(e) t^m.$$

Furthermore, the coefficient of t^n in the product is

$$A_n(e) = \sum_{j=0}^{n} a_j\, b_{n-j}(e)$$

and, because

$$\int_0^{\pi/2} \sin^{2n} x \; \mathrm{d}x = \frac{\pi}{2} \frac{\binom{2n}{n}}{4^n}$$

one obtains the formation rule

$$c_{2n}(e) = \frac{\binom{2n}{n}}{4^n} A_n(e) = \frac{\binom{2n}{n}}{4^n} \sum_{j=0}^{n} \frac{\binom{2j}{j}}{4^j} b_{n-j}(e), \qquad c_{2n+1}(e) = 0.$$

For $b_m(e)$ there is a very practical recursion law (from $(\sum b_m t^m)^2 = 1 - 4et + 4et^2$):

$$b_0 = 1, \qquad b_m(e) = \frac{r_m - \sum_{p=1}^{m-1} b_p(e) b_{m-p}(e)}{2} \quad (m \geq 1),$$

with

$$r_1 = -4e, \qquad r_2 = 4e, \qquad r_m = 0 \quad (m \geq 3).$$

And a_j likewise satisfies the simple recursion

$$a_0 = 1, \qquad a_j = \frac{2j-1}{2j} a_{j-1}.$$

Thus $c_{2n}(e)$ (and hence the g-series) can be generated systematically: first b_m by recursion, then $A_n = \sum a_j b_{n-j}$, and finally $c_{2n} = \frac{\binom{2n}{n}}{4^n} A_n$.